A Conceptual Introduction to Matrix Models of 2D Quantum Gravity

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1 2D Gravity

1.1 Motivation

The universe is not two-dimensional. In fact, gravity is quite different in two dimensions than in four-dimensions—qunatum mechanically, it describes the dynamics of scalar field. Hence, one may wonder: Why even bother studying quantum gravity?

However, qualitatively, 2D quantum gravity it is still a theory of dynamical spacetime, so one may expect that it is at least qualitatively similar to the universe in some respects, albeit far simpler. Moreover, the matrix model to be discussed enjoys some of the features believed to be present in 4D quantum gravity, and yet which seem nearly impossible to implement:

- They are discrete, without breaking symmetries associated with the continuum (e.g., Lorentz invariance).
- They are nonlocal despite have a local continuum limit.
- They consist of an emergent dimension from a large number of internal "flavor" degrees of freedom, arguably a form of holography[1].

1.2 2D Classical Gravity is Trivial

1.2.1 Trivial 2D Gravity is Trivial

Consider the linearized theory for conceptual clarity. This is more general than a mere approximation because one can always transform to a frame of reference where gravity is weak in the viccinity of a point, and then transform back to a general frame of reference. The symmetry under coordinate transformation takes the form of a gauge symmetry, which in Fourier space reads

$$h'_{ab}\left(k\right) = h_{ab}\left(k\right) + ik_{\{a}\xi_{b\}}\left(k\right), \tag{1}$$

i.e., one just arbitarrily shifts all modes with one or more index longitudinal to k_a. For freely propagating null modes, one can know right away that this kills all degrees of freedom of the metric—for such modes we know from general relativity that one impose both Lorenz and Coulomb gauge, which elimate both timelike polarizations and spacelike polarizations in the direction of propagations—i.e., *all* polarizations because there are only two directions!

This isn't quite the end of the story yet, because there may still be non-propagating, "off-shell" modes generated by sources. Consider, for simiplicity, a spacelike mode—one can always choose a frame of refrence such that $k_a = (0, k)$, and in this the metric perturbation takes the form

$$\begin{pmatrix} h_{tt} & h_{tz} \\ h_{tz} & h_{zz} \end{pmatrix}. \tag{2}$$

Note that all the components with one index in the z-direction can obviously be redefined through the gauge trasnfrmation (1). Hence, one can make the off diagonal component vanish and the zz-component equal to the minus of the tt-component; hence, in an appropriate gauge

$$h_{ab} = h_{tt}\eta_{ab}. \tag{3}$$

where the exponent garuantees positivity. Adding higher order perubations amounts to solving the linearized Einstein equation with sources generated by the stress energy of the lower order perurbations, and in general one can similarly argue that in appropriate coordinates, known as the conformal gauge, the full metric is

$$g_{ab} = \Phi\eta_{ab} \equiv e^{\phi}\eta_{ab}, \tag{4}$$

where the expoent ensures positivity. This metric simply scales all directions equally by some local factor. The problem is that the linearized Einstein equations only govern shear-like modes, and leaves source-like modes as in (3) entirely unspecificied: in two dimensions $R_{ab} - \frac{1}{2}Rg_{ab}$ is not a trace reversal, but rather a *trace elimination*, and yet the trace is the only degree of freedom for a source-like metric (4).

The problem can be traced back to the Gauss-Bonet theorem. For a source-like metric (4), the Einstein-Hilbert action turns out to take the form

$$\sqrt{g}R \propto \partial^2\phi. \tag{5}$$

This is much like computing the flux of an eletrostatic potential ψ, which will be a sum of "chrages"—the difference is that in terms of the physical metric $\psi = \log\Psi$. It is the metric which should be analytic—thus, the metric can be expanded in a "multipole" expnasion (mathemtaically, a Laurent series) around each point, $\Psi = \sum_{n=-N}^{\infty} f_n(z,t)\, r^n$. N is the order of the most violent divergence, and dominates—the "charge" contributing to (5) would then be the power of divergence, i.e., N. One thus finds

$$\int d^2x\sqrt{g}R \propto \sum_{\text{singularities}} N_s,$$

an integer result! Thus, the action has no inifinitesimal variation—it can only be varied by an integer.

1.2.2 Nontrivial 2D Gravity is Nontrivial

One way around this problem is to realize that since the metric has a preferred frame of reference where it takes the form of a scalar field, one can explicitly add dynamics to the scalar field in the usual way one adds dynamics to ordinary scalar fields. Consider a massless Klein-Gordon action for the scale factor in the conformal gauge

$$S = A\int d^2x\phi\partial^2\phi. \tag{6}$$

This can be transformed to a general gauge at the cost of locality: since $\sqrt{g}R = \partial^2\phi$, one can write $\phi = \frac{\sqrt{g}R}{\partial^2} = \frac{R}{\Box}$, where $\Box$ is the covariant d'Lambertian (which takes a simple form for a conformally flat metric). All in all, one finds

$$S = A\int \sqrt{g}d^2xR\frac{1}{\Box}R. \tag{7}$$

While this action is nonlocal, it can be generated by local degrees of freedom. Note that $\frac{1}{\Box}$ looks like the potential mediated by another massless scalar field. Indeed, $\frac{R}{\Box}$ is the solution of the equation $\Box\varphi = R$, so (7) is equivalent to the local action

$$A\int d^2x\left(\sqrt{g}\varphi R + \frac{1}{2}\eta^{ab}\partial_a\varphi\partial_b\varphi\right). \tag{8}$$

Eq. (8) is known as the Liouville action. The same trick of expressing the nonlocal action (7) as a local action works in the quantum theory—instead of solving for φ, one has to integrate φ out, but the result will be the same.

1.3 2D Qunatum Gravity is Nontrivial

It turns out that quantum mechanics generates the Liouville action (8) naturally.

General covariance allows one to transform to the conformal gauge where the metric takes the form of a scalar field (4). It was shown that the Einstein equation leaves the scalar field arbitrary. A fancy way to say it is that one has an additional symmetry under shifting that scalar—i.e., under rescaling the metric by a local factor,

$$g_{ab} \to e^{w(x,t)}g_{ab}. \tag{9}$$

This is known as a *Weyl symmetry*.

While Weyl symmetry is respected by the action, it is broken by the usual measure $\mathcal{D}g$, because it does depend on the scale factor ϕ. This will give rise to dynamics—in this sense, two dimensional gravity is, in a sense, an entropic force—it arises not due to differences in classical action but due to the different quantum weights of classical configurations!

To see that the measure breaks Weyl symmetry, note that a general metric is just the metric in the coordinates where it takes the scalar form, transformed general coordinates. The infinitesimal form of an arbitrary coordinate transformation from conformal gauge is $g'_{ab} = \Phi\eta_{ab} + \nabla_{\{a}\delta\xi_{b\}}$. This gives the variation of the metric

$$\delta g_{ab} = \delta\Phi\eta_{ab} + \nabla_{\{a}\delta\xi_{b\}}. \tag{10}$$

To see how the measure changes under a shift of Φ, one needs some way of determining how big this variation is. To this end, I define a "metric of the space of metrics," treating the metric variation at each point as different orthogonal directions in the "space." Thus, one has to sum over quadratic invariants of the metric at different points with respect to the invariant volume element $\sqrt{g}d^2x$. The most general such sum is, up to a constant factor,

$$||\delta g||^2 = \int \sqrt{g}d^2x\left(g^{ab}g^{cd} + Cg^{ac}g^{bd}\right)\delta g_{ac}\delta g_{bd}. \tag{11}$$

The coordinate transformation can be decomposed into a scaling part $\nabla_b \delta\xi^b g_{ab} \equiv \delta\Omega$ and a shearing part $\delta S_{ab} = \nabla_{\{a}\delta\xi_{b\}} - \nabla_b \delta\xi^b g_{ab}$. Unsurprisingly, the scaling can be cancelled by a shift of the scale factor. Indeed, plugging the variation of the metric (10), one obtains

$$||\delta g||^2 = \int \sqrt{g} d^2x \left[(1+2C) \left(\delta\phi + \delta\Omega \right)^2 + \delta S^{ab} \delta S_{ab} \right] \tag{12}$$

$$\equiv \int \sqrt{g} d^2x \left[(1+2C) \delta\tilde{\phi}^2 + \delta S^{ab} \delta S_{ab} \right]. \tag{13}$$

The shear term $\delta S^{ab}\delta S_{ab}$ can is quadratic in ξ_a, and by integration by parts can be rewritten as

$$\begin{aligned} \delta S^{ab}\delta S_{ab} &= \delta\xi^b \nabla^a \left[\nabla_{\{a}\delta\xi_{b\}} - \nabla_b \delta\xi^b g_{ab} \right] + \text{boundary term} \\ &\equiv \delta\xi^a \mathcal{O}_a^b \delta\xi_b, \end{aligned}$$

so that the "volume element" of the path integral transforms as the determinant of the transformation:

$$\mathcal{D}g = \sqrt{\det \mathcal{O}} \mathcal{D}\phi \mathcal{D}\xi, \tag{14}$$

where the determinant is product of eigenvalues of $\mathcal{O}$, satisfying in conformal gauge

$$e^{-2\phi} \eta^{ab} \partial_a \left(e^{\phi} \partial_b \psi_n \right) = \lambda_n \psi_n, \tag{15}$$

which can be solved, e.g., by the usual eigenvalue perturbation series in ϕ as a "potential" for the "wavefunction" ψ. The dependence of the measure on the scale factor signals the breakdown of Weyl symmetry.

Now, I argue that there is a unique way to quantize gravity preserving Weyl invariance—by modifying the classical action. The idea is following: start from a given configuration of the metric, say, flat spacetime. Suppose the action is, for instance, zero (as it would be for the Einstien-Hilbert action). Because of the scalar nature of the 2D metric, a general configuration can be generated by Weyl transformation and a coordinate transformation (the latter of which should not affect the action). I demand that the integrand times the measure $\mathcal{D}g e^{-S}$ remains unchanged, i.e.,

$$\mathcal{D}g \left[g_{ab} \right] e^{-S[g_{ab}]} \equiv \mathcal{D}g \left[e^w \eta_{ab} \right] e^{-S\left[e^W \eta_{ab} \right]} = \mathcal{D}g \left[\eta_{ab} \right]. \tag{16}$$

Since $\mathcal{D}g$ changed as $\sqrt{\det \mathcal{O}}$, S must be given by

$$S \left[g_{ab} \right] = \frac{1}{2} \log \det \mathcal{O} \left[g_{ab} \right] = \frac{1}{2} \mathrm{Tr} \log \mathcal{O} \left[g_{ab} \right]. \tag{17}$$

Now, computing $\mathrm{Tr}\mathcal{O}\left[g_{ab} \right]$ in detail is beyond the scope of the lecture. Let me just point out that it is no miracle that it take the form of the Liouville action: the variation of the action with respect the metric, is, by definition, the stress-energy tensor, and the variation of the action with respect to the scale factor is its trace. Generally, it is covariant operator the scales as $1/\ell^2$, so it clearly only involves second derivatives of the metric, and only a linear power of the second-order derivative. Now, because the metric has the scalar form (4), it is not surprising that it has only one second-order invariant—it is just the Ricci scalar, or in conformal gauge $e^{-\phi}\partial^2\phi$. This immediately gives rise to the Liouville action. For pure gravity, the prefactor turns out to be

$$A = \frac{13}{24\pi}.$$

Adding D massless scalars changes A to

$$A = \frac{26 - D}{48}.$$

To cure the scale anomaly, the scale factor φ has been turned into a dynamical variable. This is similar to the way in which one can, say, restore gauge symmetry to a massive vector field by turning the transverse mode into an invariant dynamical variable, which is how Stueckelberg discovered an early version of the Higgs mechanism.

1.4 2D String Theory is 2D Quantum Gravity

D-dimensional string theory can be regarded as a theory of many fields living on an internal *2-dimensional worldsheet*, generating a *D-dimensional fields in the low-energy limit*. Hence, 2-dimensional string theory can be regarded as a theory of 2-dimensional wordlsheet fields with 2-dimensional spacetime fields low-energy limit—i.e., a rather standard two-diemsnional field theory! Of course, one unique characteristic compared to other field theory is that the background worldsheet metric acquires dynamical corrections from the fields; this means that two-dimensinoal string theory is just a field theory of two-dimensional quantum gravity, and, in fact, it turns out to be exactly equivalent to the Liouville theory discussed above. In fact, I believe this is the context in which the Liouville action was originally proposed[2].

Indeed, the action (8) looks much like a standard string action, with the exception of the Ricci term. It can be given two different interpretations: For one, it can be interpreted as the action of a one-dimensional string in Minkowski background, but due to Weyl anomaly, the dilaton becomes a fully dynamical field, in the same way as the scale factor became a fully dynamical variable in section 1.3 above. A different interpretation is that of a string propagating in two dimensions, where the dilaton (i.e., string coupling) is not dynamical but determined nontrivially by the string coordinates:

$$\varphi(\tau, \sigma) = \sqrt{\frac{26 - D}{6\alpha'}} X^1.$$

Now, string theory in two dimensions is remarkably weird, or rather, remarkably unweird! It contains essentially none of the complex structure of higher dimensional string theory, with the exception of gravity. To see this, try to picture a string moving in a 1-dimensional space (i.e., 1+1 dimensional spacetime). Evidently, the string is just a line on a line—it cannot deform, but only move from one point to the next; hence, it essentially behaves like a point particles, and many strings that can be created or annihilated behave like a normal field!

Now, this can shown in a greater mathematical. In general, a string can be viewed as a tower of particles: the state of a single string can be written as a superposition of excitations

$$\mid \Psi\rangle = \left(A_0 + A_1 e_i^1 \alpha_{-1}^i + A_2 e_i^2 \alpha_{-2}^i + A_3 e_{ij}^3 \alpha_{-1}^i \alpha_{-1}^j + ...\right) \mid 0, \boldsymbol{k}\rangle, \quad (18)$$

where i and j are transverse spatial indices

$$i, j = 1, 2, ...D - 2. \quad (19)$$

However, for $D = 2$, $D = 2 = 0$! There is only one spatial direction, and thereby no transverse direction through which the string can vibrate. Hence, a string can either be at the ground state or not be at all. The tower of particles is thus truncated at its first floor.

In higher dimensional string theory, the ground state has a negative mass squared. This is a problem, because it means an infinite number of strings will be produced at any finite energy, releasing an infinite amount of energy. However, in two dimensions this is not the case. The only degeree of freedom of the ground state are its momentum, and it ressponds to the plane wave vertex operator $e^{ik\cdot X}$, which one adds as a perturbation to the action. The stress-energy generates reparametrizations, and the scaling dimension Δ can be read off from the operator product expansion with the-stress energy:

$$\mathcal{T}\left(T\left(0\right)e^{ik\cdot X(z)}\right)=\frac{\Delta}{z^2}e^{ik\cdot X(0)}+\mathcal{O}\left(\frac{1}{z}\right).$$

This stress tensor is not quite the usual stress tensor of string theory—it is obtained by varying the action (8) with respect to the metric, and gives

$$T\left(z\right)=-\frac{1}{2}:\partial X^{\mu}\partial X_{\mu}:+\sqrt{\frac{26-D}{24\alpha'}}\partial^2X^1. \tag{20}$$

Scale invariance requires that $\Delta=\tilde{\Delta}=1$ (like the scaling of the usual Lagrangian. This gives

$$\left(k^{\mu}+i\sqrt{\frac{26-D}{6\alpha'}}\delta_1^{\mu}\right)^2=-\frac{26-D}{6\alpha'}-\frac{4}{\alpha'}.$$

For $D=2$, one finds that the ground state is massless.

2 "Pure" Matrix Models

2.1 The Sum Over Triangulations

One can make sense of the sum over surfaces by tiling space with discrete triangles. This gets rid of UV divergences, whose origin is in aribrarily small lengthscales. This also makes calcuations, in principle, numerically tractable, because it turns the continuous path integral into an algebraic sum. I assume that all triangles are equilateral and of the same size—there is no dynamical "intrinsic length," "intrinsic area" or "instrinsic angle" to a triangle—all geometry follows from the structure of the triangulation.

To be more specific, the Riemann tensor is defined as follows: R^a_{bcd} is the angle deficit per unit area in a-direction of a vector in d-direction, parallel-trasported along the bc plane. For conformally flat spacetime, everything is spherically symmetric, so the Ricci scalar is proportional; hence, one has at the ith vertex

$$R_i=2\frac{\Delta\theta_i}{A_i}. \tag{21}$$

Now, if there is a vertex where N triangles intersect, since each triangle contributes an angle of $\frac{\pi}{3}$ (as expetced from equilateral triangles) the total angle around the loop is $\frac{N\pi}{3}$. In flat space, one would expect 2π. Moreover, single each triangle contributes, say, a unit area, but has three different vertices, so to avoid triple counting only a third of its area must be associated with each particular vertices. Hence, the combined area is $N_i/3$. All in all, one has

$$R_i=2\pi\frac{6-N_i}{N_i}. \tag{22}$$

The squre root of the determinant of the metric is just the discrete area element $\frac{N_i}{3}$. All in all, the discretized Einstein-Hilbert action is

$$S = \sum_i \sqrt{g_i} R_i = \sum_i 2\pi \left(2 - N_i/3\right). \tag{23}$$

Likewise, one can add a cosmological constant term

$$S_\Lambda = \sum_i \Lambda \sqrt{g_i} = \sum_i \Lambda \frac{N_i}{3}. \tag{24}$$

Combining both terms, the discretized path integral can be written as

$$\sum_{\text{triangulations}} \exp\left[-\sum_i \left\{2\pi \left(2 - N_i/3\right) + \Lambda N_i/3\right\}\right]. \tag{25}$$

2.2 The Matrix Dual

Perhaps the most important insight to take from this lecture is that the structure of surface triangulations can be uniquely mapped to "bound states" of matrices M_{ij}. Why matrices? Because a matrix has two "flavor charges," i and j, and its Feynman diagram can thus be represented as a double line; each flavor can be contracted with another line while leaving the other flavor free. If each contraction is drawn as merger of one line with another, this means that a given matrix can be connected to an arbitrary number of lines. This has to with the fact that while vectors only have one invariant, $\vec{v} \cdot \vec{v}$, matrices have an infinite number of them: $\mathrm{Tr}\left(\boldsymbol{M}\right)$, $\mathrm{Tr}\left(\boldsymbol{M}^2\right)$, $\mathrm{Tr}\left(\boldsymbol{M}^3\right)$, etc.

So how does one identify a Feynman diagram of marices with a triangulation? Well, each matrix propagator, which is, again, represented by two lines, is uniquely mapped to a third perpendicular to both lines. Hence, a cubic vertex is equivalent to a triangle.

All in all, one has the cubic matrix integral

$$Z = \int dM \exp\left(-\frac{1}{2}\mathrm{Tr}\left(\boldsymbol{M^2}\right) - g\mathrm{Tr}\big(\boldsymbol{M^3}\big)\right) = \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \left\langle \mathrm{Tr}\left(\boldsymbol{M}^3\right)^n \right\rangle_0, \tag{26}$$

where $\langle\cdot\rangle_0$ denotes averaging over the Gaussina part of the measure. Note that the expansion in powers of the coupling loooks like the expansion in powers of $e^{-\Lambda/3}$. Hence, one identifies its exponential with the cosmological constant

$$g \sim e^{-\Lambda/3}, \tag{27}$$

which becomes exact when one works out the sum over connected diagrams $\ln Z$ and keeps track of factors, which I'm not going to. Indeed, physically, the coupling determines the porbability of adding an extra vertices—just like the cosmological constant the energy cost of exrta points (i.e., volume).

2.3 The Large N Limit and the Continuum Limits

The continuum theory is characterized by varialbes at each of the infinite number of points, variables which can take an infinite number of possible values. Hence, in the continuum limit one expects both the nubmer of vertices to diverge number of vertex *types* to diverge. The former corresponds to a crtitical points with an infinite "correlation length"—the latter corresponds to a large number of flavors for the matrices.

Note, first, that the effective coupling is in fact gN—there are N ways through which an interaction can take place due to exchange of two flavors, and the more ways there are the likelier will interaction be. Hence, unless one wants a strongly coupled limit, it is gN—not g—that one should keep constant in the limit $N \to \infty$. gN is known as *'t Hooft coupling*. Further, due to the growing number of matrix entries, the overall action tends to scale as N^2. Hence, the large N limit has a self-averaging effect—it represses fluctuations, and becomes clalssical, much like a mean-field theory iin statistical mechanics. Because of the scaling of the action, it is convenient to redefine it in terms of variables that don't scale, by setting

$$\boldsymbol{M} \to N\boldsymbol{M} \tag{28}$$

$$g \to \frac{g}{N}, \tag{29}$$

in which case the action takes the form

$$S = N^2 \left(\frac{1}{2} \mathrm{Tr} \left(\boldsymbol{M^2} \right) + g \mathrm{Tr} \big(\boldsymbol{M^3} \big) \right) \equiv \beta \left(\frac{1}{2} \mathrm{Tr} \left(\boldsymbol{M^2} \right) + g \mathrm{Tr} \big(\boldsymbol{M^3} \big) \right), \tag{30}$$

where $\beta \equiv N^2$ plays the role of an inverse temperature in that when N^2 is large, only the minimum of the action dominates.

How do different diagrams scale with N^2? Well, each pair of lines, i.e., edge, contributes a factor of $\frac{1}{\beta}$ as the inverse of the quadratic term (fluctuations scale as temperature); each vertex contributes a factor of $g\beta$ due to the the effective coupling; each closed loop, i.e., face contributes a factor of β due the summation over all possible entries. All in all, one has

$$A\left(V, E, F\right)] \sim \beta^{V-E+F} = \beta^{\xi}. \tag{31}$$

ξ is the Euler characteristic. A Stackexchange answer explains conceptually why it is a topological invariant[3]: It depends only on the surface and not on the speicifc triangulation. One can add a vertex on a line, dividing the line into two. This adds a vertex and an extra line, and thus leaves the Euler number unchanged. Likewise, for a given shape, one can send a line from one vertex to the next. This adds a line and an extra face, and thus, again, cancels. You can easily see that a torus has an Euler characteristic of zero: it is just a square with each two opposite edges identified, so it has two edges, a single vertex, and a single face. A torus has a single hole. How does one extend the result to two holes, i.e., a double torus? By connecting two tori by a cylinder. This can be done by adding an extra edge to each torus going through the hole and parallel to the other, pre-existing edge (the one from the "square with opposite edges identified" argument) that does the same, then dividing the cylinder into two squares going all through the axis of symmetry, squares which intersect with each torus's aforementioned two edges. This is very hard to picture, but after some notorious trial and error one finds that this adds six faces, six vertices and 14 lines—i.e., an Euler characteristic of minus two. Then one can use this inductively a shape with any number of holes—e.g., a triple torus. All in all, one finds

$$\xi = 2 - 2g, \tag{32}$$

with g the number of holes. Hence, the low temperature expansion looks like the sum over topologies from string theory, and β is identified as g_{string}. Hence, one might expect that the large N

(i.e., large β) limit is just weakly coupled, containing nothing but the trivial topology of a sphere. However, it will be seen that at the critical point where the number of triangles diverges it so happens that the *coefficients* of high loop amplitudes diverge in a way that cancels the vanishing of β^{ξ}. Taking the large-N limit and going to the critical point simulatenously is known as the double scaling limit.

2.4 The Haar Measure and the Classical Limit

One computes the matrix integral (26) in the same way as one computes a volume integral—the first step is to exploit symmetries, that reveal certain variables of which the integrand is independent. For a volume integral, one often exploits spherical symmetry. In the matrix integral, one uses the similar $U(N)$ rotational symmetry—just like one can decompose a vector into a magnitude and a direction (which is not interesting in cases of spherical symmetry), one can decompose the matrix into a diagonal part and a unitary rotation which is not very interesting because of the $U(N)$ symmetry:

$$\boldsymbol{M} = \boldsymbol{U}^{\dagger}\boldsymbol{\Lambda}\boldsymbol{U}. \tag{33}$$

Hence, one can replace the integral over matrices with a simpler integral over eigenvalues. However, there is a caveat: not all eigenvalue configurations are created equal. There is a nontrivial measure for the integration, known in general as the Haar measure. The measure of the integration over matrices should be the same in different bases, much like the measure of one-dimensional integration dx is the same at different points. In particular, one can compute in the diagonal basis, and the result will generalize to higher dimensions. In the diagonal basis, one has $\boldsymbol{M} = \boldsymbol{\Lambda}$. Its differential is the sum of a shift of the eigenvalues and an infitesimal basis rotation (which, as usual, is a commutator)

$$d\boldsymbol{M} = d\boldsymbol{\Lambda} + [d\boldsymbol{U}, \boldsymbol{\Lambda}], \tag{34}$$

and in explicit notation

$$dM_{ij} = \delta_{ij} d\lambda_i + (\lambda_i - \lambda_j)\, dU_{ij} \text{ (no summation over recurring indices).} \tag{35}$$

It is seen that there is an effective repulsion: if two eigenvalues are nearly degenerate, their block looks nearly like an indentity matrix—one will need a very fine-tuned rotation to make this matrix look different from an identity matrix, and thus there are fewer such matrices. The "volume" element transforms as the determinant of the transformation of the "line" element; hence, the measure is

$$dM = \Delta^2(\lambda) \left(\prod_i d\lambda_i\right) \left(\prod_{ij} dU_{ij}\right), \tag{36}$$

where

$$\Delta(\lambda) = \prod_{i \neq j} (\lambda_i - \lambda_j), \tag{37}$$

known as the Vandermonde determinant.

As mentioned above, the large N limit is a classical limit, given by a saddle point approximation. Naively, one this think would this places all eigenvalues at the minimum of the potential $\frac{1}{2}\lambda^2 - g\lambda^3$. However, in the double scaling limit one zooms into a very particular point, so one cannot assume that repulsion is negligible. The configuration of stationary phase is instead the minimum of "free energy," that has both an entropic contribution and a contribution from the action

$$F = \beta^{-1} (\log \Delta - S) = \beta^{-1} \sum_{i \neq j} \log (\lambda_i - \lambda_j) - \sum_i \left(\frac{1}{2} \lambda_i^2 + g \lambda_i^3 \right). \tag{38}$$

The eigenvalues thus behave like charges with electrostatic repulsion, put in the external potential $-\left(\frac{1}{2} \lambda^2 + g \lambda^3 \right)$—they tend to set at the minimum but cannot get overly close.

3 Propagating Matrix Models

3.1 Path Integral Picture

Liouville gravity (like 2D string theory) is not merely given by the Einstein-Hilbert—in addition, there is the scalar field. A discretized scalar field in one dimension X has the kinetic term which an energy cost for variation among nearest neighbors.

$$T = \begin{cases} \frac{(X_I - X_J)^2}{2} & I, J \text{ nearest neighbors} \\ 0 & \text{otherwise} \end{cases}.$$

So the pertubration gets terms like $\prod_{\langle IJ \rangle} \int dx_I e^{-\frac{1}{2}(x_I - x_J)^2}$. Saying that x propagates along the surfaces generated by the matrix is equivalent to the statement that M propagates along the possible values of X—it is a mere shift of passve-to-active perspective. Hence, one adds a kinetic term to the matrix. This kinetic term is the operator inverse of the propagator $e^{-\frac{1}{2}(x_I - x_J)^2}$. In the continuum limit, the kinetic term approaches a derivative, and one has the path integral

$$Z = \int \mathcal{D} M \exp \left\{ -\beta \int \operatorname{Tr} \left[\frac{1}{2} (\partial_x \boldsymbol{M})^2 - \frac{1}{2} m^2 \boldsymbol{M}^2 - g \boldsymbol{M}^3 \right] dx \right\}. \tag{39}$$

3.2 Schrodinger Picture and Free Fermions

Treating x as time rather than a time coordinate, one can readily extract the Hamiltonian from the action. Like in quantum mechanics of particles, the energy corresponds to a hopping-like Laplacian coupling amplitudes at neighboring points. In section 2.4, I computed the matrix line element in terms of eigenvalues and basis transformations, from which the transformation of the Laplacian can be computed. The result is the sum of a "rotational kientic energy" and a "radial kinetic energy":

$$E = -\frac{1}{2\beta^2 \Delta(\lambda)} \sum_i \frac{\partial^2}{\partial \lambda_i^2} \Delta(\lambda) + \sum_i U(\lambda_i) + \text{rotational energy}. \tag{40}$$

β^{-2} is the analogue of $\frac{\hbar^2}{2m^2}$. The radial kinetic energy is the analogoue of $\frac{\hbar^2}{2m} \frac{1}{r} \partial_r^2 (r\psi)$. Because there is no angular potential energy, one can safely assume that at the ground state of the system, rotational energy vanishes. This implies not merely that Ψ is independent of U_{ij}, but further that it is symmetric eigenvalue permutations since such permutations are rotations by $\pi/2$.

The Vandermont determinant generates repulsion betweeen eigenvalues. However, one can multiply both sides of (4) by the determinant to obtain

$$E\Delta(\lambda)\Psi = \left(-\frac{1}{2\beta^2}\sum_i \frac{\partial^2}{\partial\lambda_i^2} - \sum_i \frac{m^2}{2}\lambda_i^2 - \sum_i g\lambda_i^3\right)\Delta(\lambda)\Psi. \tag{41}$$

At the ground state, Ψ is presumably symmetric. Because $\Delta(\lambda)$ is antisymmetric, the effective wavefunction $\Delta(\lambda)\Psi$. Hence, the repulsion between eigenvalues has been eliminated at the cost of turning it into a Pauli exclusion! The effective wavefunction $\Delta(\lambda)\Psi$ has a simple interpreation: its square modulus

$$P(\lambda) = \Delta^2(\lambda)\left|\Psi(\lambda)\right|^2 \tag{42}$$

is just the probabiltiy density multiplied by the volume element of matrices—i.e., the probability per-unit eigenvalue.

3.3 Critical Behavior

The fermionic nature of the ground state sheds light on the nature of the transition: assume that one starts at the meta-stable minimum of the potential. To add fermions, one simply fills the lowest-lying available states in its valley. At some point, fermions cross the barrier and spill into the other side of the potential *ad-infinitium*. For a given number of eigenvalues (i.e., for a given size of the matrices), whether or not the barrier is crossed is determined by the height of the barrier—namely, by the coupling g.

Clearly, there is *a* phase transition at some critical g, but is it *the* phase transition that corresponds to the continuum limit? In fact, it is. Since the n triangle contribution scales as g^n, g can be thought of as a "generator of triangles"—$\frac{\partial}{\partial g}$ acting on g^n gives ng^{n-1}; the sum of all such terms weighed by the amplitude, i.e., $\frac{\partial Z}{\partial g}$ hence gives the sum of n-triangle amplitudes multiplied by $\frac{n}{g}$—that is, the expected number of of triangles. The normalized expectation is obtained by dividing by the normalization

$$\langle n\rangle = \frac{\partial \log Z(g)}{\partial \log g} = \frac{g}{Z}\frac{\partial Z}{\partial g}. \tag{43}$$

A small change of g has the effect of changing the height of the barrier. It is clear that there is a discontinuity in the partition function when the barrier becomes low enough for fermions at the Fermi level to pass through—hence a divergence in the number of triangles.

Shifting the coupling in such a way that the height of the barrier is equal to the Fermi-level is equivalent to shifting the Fermi level to be the height of the barrier. Hence, to study crticial behavior one expands in λ for small deviations from the height of the barrier

$$\bar{\lambda} = \gamma(\lambda - \lambda_c); \tag{44}$$

$$\bar{E} = E - E_c, \tag{45}$$

where E_c is the potential at the top of the barrier, λ_c is the position of the barrier, and γ is a scaling factor. Note that in the double scaling limit, the classical limit $\beta \to \infty$ and the continuum limit $\gamma \to \infty$ are taken independently. Hence, the scaling of λ must be such that β remains constnat—since β^{-1} (or $\hbar$) is "the minimum uncertainty of position times uncertainty of momentum,"

momentum must scale as γ^{-1}. Hence, noting that in the way in which the Hamiltonian was defined the maximum is already centered at $\lambda_c = 0$ with potential energy $E_c = 0$, the scaled Hamiltonian, expanded around the maximum, takes the form

$$\bar{E}\Psi_{\text{eff}} = \left(-\frac{\gamma^2}{2\beta^2}\sum_i \frac{\partial^2}{\partial\bar{\lambda}_i^2} - \frac{m^2}{2\gamma^2}\sum_i \bar{\lambda}_i^2 - \frac{g}{\gamma^3}\sum_i \bar{\lambda}_i^3\right)\Psi_{\text{eff}}. \tag{46}$$

Evidently, the cubic term scales to zero as $\gamma \to \infty$. This makes sense—one is only interested in the vicinity of the barrier in the infinite limit, which involves quadratic—higher order terms are at least "small times small times small" around the maximum. This is at the core of the exact solvablity of the matrix model.

Now, we know from the harmonic oscillator that that low-lying energies scale as $\hbar\omega$, or

$$E \sim \beta^{-1} m. \tag{47}$$

Typical variations of quantities on the right hand side of (46) generate small energy changes only insofar as a the right hand-side has the same scaling, assuming all positions and momenta are of order unity. That is, one has $-\frac{\gamma^2}{2\beta^2}\sum_i \frac{\partial^2}{\partial\lambda_i^2} + \frac{m^2}{2\gamma^2}\sum_i \bar{\lambda}_i^2 \sim \beta^{-1}$, which implies

$$\gamma \sim \beta^{-1}. \tag{48}$$

Eq. (47) then takes the much simplified form

$$\bar{E}\Psi_{\text{eff}} = \frac{1}{2\beta}\sum_i \left(-\frac{\partial^2}{\partial\bar{\lambda}_i^2} - \bar{\lambda}_i^2 + \mathcal{O}\left(\frac{1}{\sqrt{\beta}}\right)\right)\Psi_{\text{eff}}. \tag{49}$$

This is an equation for an inverted harmonic oscillator.

With this in mind, one can readily compute the density of energy eigenvalues. $\delta\left(\hat{H} - \mu\right) = \text{diag}\delta\left(E_n - \mu\right)$ just gives $\delta\left(0\right)$ for entries where μ coincides with the energy levels and 0 otherwise. Hence, summing over all entires

$$\rho\left(\mu\right) = \text{Tr}\delta\left(\hat{H} - \mu\right) = \frac{1}{\beta}\text{Tr}\delta\left(\beta\hat{H} - \beta\mu\right) \tag{50}$$

is just the number of such eigenvalues. One can express the delta function as a sum over destructively interfering plane waves

$$m\rho\left(\mu\right) = \frac{1}{2\pi\beta}\int_{-\infty}^{\infty} d\tau e^{-i\left(\beta\hat{H} - \beta\mu\right)\tau} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau e^{-i\beta\hat{H}\tau} e^{i\beta\mu\tau},$$

and $e^{i\hat{H}T}$ is just the propagator. For the usual harmonic oscillator, the propagator is the familiar

$$\begin{aligned}\left\langle x_f \left| e^{i\hat{\mathcal{H}}_{\text{harmonic}}\tau} \right| x_i \right\rangle &= \int_{x_i}^{x_f} \mathcal{D}X \exp\left[i\frac{1}{2}\int_0^\tau dt\left(\dot{x}^2\left(t\right) - x^2\left(t\right)\right)\right] \\ &= \left(\frac{1}{i\pi}\right)^{1/2} \exp\left[-\frac{\left(x_i^2 + x_f^2\right)\cos\left(\tau\right) - 2x_i x_f}{i\sin\left(\tau\right)}\right].\end{aligned}$$

To convert this to the inverted harmonic oscillator, all one has to do is switch to imaginary frequency (which turns oscillations into exponential growth)—i.e., to transform to imaginay time. This gives

$$\rho(\mu) = i\int_{-\infty}^{\infty} d\tau e^{-i\beta\mu\tau}\int_{-\infty}^{\infty} dx \left\langle x \left| e^{-\hat{\mathcal{H}}_{\text{harmonic}}\tau} \right| x \right\rangle \tag{51}$$

$$= i\int_{-\infty}^{\infty} d\tau e^{-i\beta\mu\tau}\sqrt{\frac{i}{\pi\sinh(\tau)}}\int_{-\infty}^{\infty} dx \exp\left(2\frac{\cosh(\tau)-1}{\sinh(\tau)}x^2\right). \tag{52}$$

The integral over x is Gaussian and can be readily evaluated, but the integral over τ doesn't appear particularly convergnet—it scales as $\frac{e^{-i\mu\tau}}{\sqrt{\cosh(\tau)-1}} \sim \frac{e^{-i\mu\tau}}{\tau}$ for small times. This is not surprising—the inverted harmonic oscillator is unstable, so it involves many frequencies—anyway, the integral (51-52) is invalid for large enough energy fluctuations, where the particle is expetced to be away from the top of the barrier. However, one should know that the derivative of the Forurier transform with respect to frequency has a softer short-time behavior than the Fourier transform itself. Indeed, $\frac{\partial}{\partial\omega}$ is equivalent to $-it$, which would exactly cancel the linear divergence. Hence, one can differentiate (52) with repsect to μ, expand the resulting the integral in a power series, and then reintegrate the series. The result is

$$\rho(\mu) = \frac{1}{2\pi}\left(-\ln\mu + \sum_{m=1}^{\infty}\left(2^{2m-1}-1\right)\frac{|B_{2m}|}{n}(2\beta\mu)^{-2m}\right) + \text{const.}$$

Here, B_m are the Bernouli numbers, defined through the recursion

$$B_0 = 1;\; B_m = -\frac{1}{m+1}\sum_{i=0}^{m-1}\binom{m+1}{i}B_{i.}$$

The constant is infinite for the inverted harmonic potential (and presumably forr the cubic ptential), but I suppose it should be finite for potentials that make more sense far from the top of the barrier, and which thus have a spectrum similar the one of a harmonic oscillator or of a free particle depending of whether states are asymptotically bound or freely propagating. At at any rate, one can use the density of eigenvalues to obtain, say, the ground state energy, which would be given by $\int_{-E_{F-}}^{0} d\mu\rho(\mu)$. The result is a power series in $\beta\mu$, much like the sum over topolgies in string theory.

It is now seen that one can take the continuum limit $\mu \to 0$ and the continuum limit $\beta \to \infty$ simulatenously and the sum still contains all topologies—i.e., all powers of β—as was claimed in section 2.3. The reason can be physically interpreted—the classical limit is effectively a low temperature limit, while the continuum limit is the limit of small energy differences from the top of the barrier. Clearly, what determines the probability for a fluctuation is the energy relative to the thermal energy, $\frac{E}{T}$, i.e., $\beta\mu$—thus, focusing on small energies while reducing thermal fluctuations has a cancelling effect.

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