A Conceptual Introduction to KPZ Surface Growth Model

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1 Universality of Growing Surfaces

Consider the following three systems:

1. deposition of a gas into a crystal;
2. spatial growth of a bacterial colony;
3. burning of a sheet of paper (and presumably a forest).

At first glance, these systems do not seem to share much in common—the mircoscopic mechanisms by which they are governed are entirely different. And yet, it turns out that their macroscopic behavior shares just about everything in common:

1. All of these systems are far from thermal equilibrium. By itself, this doesn't say much—it says what these systems *aren't*, but not what they *are*. After all, far from equilibrium behavior is much more varied than its equilibrium counterpart.
2. In all three cases, points irreversibly "infect" neighboring poitnts with some property (crystalization, bacterial population, burning). Hence, only degrees of freedom at the very surface of the infected region are dynamical—bulk degrees of freedom cannot become infected since they already are, and have nothing to infect since their neighbors are infected.
3. All of these systems are experimentally found to be characterized by the same scale-invariant spread of perturbations to a distance $\delta x \sim \delta t^{2/3}$—much faster than diffusion ($\delta x \sim \delta t^{1/2}$).

That the spread of fluctuations is scale invariant suggests that mircoscopic details are immaterial—they can simply be scaled away. Scaling microscopic details away, one is left with a smeared continuum theory—*a field theory*. That the spread of perturbations is the same in all of the above systems suggests that the field theory is *universal*. While there are crucial small-scale differneces between these models, all of them fade away when one takes off one's glasses (or microscope).

However, note that such a scaling cannot be reproduced by any linear field theory. Over long times, since it takes perturbations a finite time to relax when subject to any arbitarily small drag, one expects the long-term motion of the surface to be governed by the terminal velocity:

$$\frac{\partial h}{\partial t} = b^{-1} F,$$

with h the height at a given point, and F the forces acting on the surface. Now, assuming the forces are linear and translationally invariant, the equation should be the overdamped Newton law for decoupled oscilalators:

$$\frac{\partial h_{\boldsymbol{q}}}{\partial t} = -K\left(\boldsymbol{q}\right) h_{\boldsymbol{q}} + \eta\left(\boldsymbol{q}, t\right), \tag{1}$$

with $K\left(\boldsymbol{q}\right)$ the "spring constant" of the qth mode, and $\eta\left(\boldsymbol{q}, t\right)$ a Gaussian, uncorrelateed noise

$$\left\langle \eta\left(\boldsymbol{q}, t\right) \eta\left(\boldsymbol{q}', t\right) \right\rangle = D\delta_{\boldsymbol{q},\boldsymbol{q}'}, \delta\left(t - t'\right).$$

The solution is straightforward: one finds for the fluctuations

$$\left\langle h\left(\boldsymbol{q}', t'\right) h\left(\boldsymbol{q}, t\right) \right\rangle = \frac{D\delta_{\boldsymbol{q},-\boldsymbol{q}'}}{K\left(\boldsymbol{q}\right)} \exp\left(-K\left(\boldsymbol{q}\right)\left|t - t'\right|\right).$$

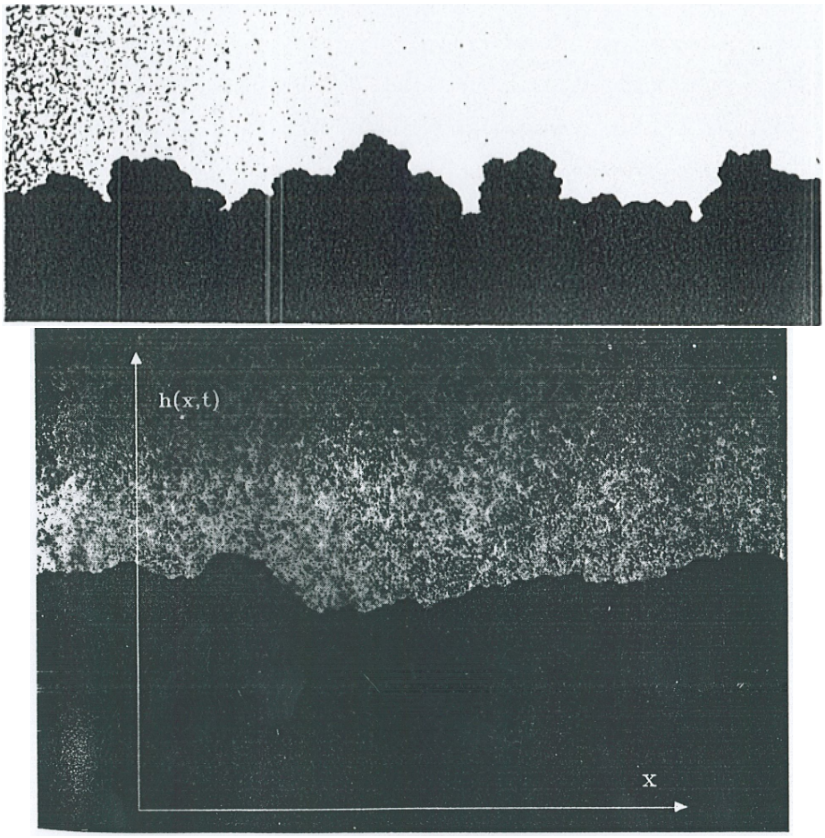

Figure 1: Top: Surface profile of a bacterial colony from [1]. Bottom: Surface profile of a burning sheet of papaer from [2].

This result is easy to interpret: modes with larger K are harder to excite and thus relax faster. Decay of small wavelength modes in Fourier space is spreading in real space, and so it is seen that the time taken for fluctutaions to spread is

$$\delta t \sim K\left(\frac{1}{\delta x}\right).$$

In order to have the right scaling, one must set

$$K\left(\boldsymbol{q}\right) = C\left|\boldsymbol{q}\right|^{2/3}, \tag{2}$$

but in real space $\boldsymbol{q} \to -i\boldsymbol{\nabla}$, and this means that eq. (1) should become highly nonlocal. Therefore, if locality is to preserved, this scaling entails nonlinearity.

What kind of a nonlinear can describe these systems? In experiment, the growth pattern never becomes quite smooth—rather, each narrow kink becomes a parabola-like curve getting wider and wider, overrunning any smaller kinks. Experimental profiles of flame and bacterial colony surfaces are shown in figure 1. It is known that a diffusion equation turns a kink into a Gaussian getting wider and wider—a prabola is just the logarithm of a Gaussian, and so one may expect the equation to be the logarithm of a diffusion equation.

2 The KPZ Equation

The idea of KPZ[3] was to add to the noise-driven harmonic oscillator equation a-la (1) a growh velocity, $v\left(h, \boldsymbol{\nabla} h, ...\right)$:

$$\frac{\partial h}{\partial t} = -\nu \nabla^2 h + v_{\text{growth}}\left(h, \boldsymbol{\nabla} h, ...\right) + \eta\left(\boldsymbol{x}, t\right). \tag{3}$$

The first term is the usual diffusion term, which has the effect of smoothening fluctutations. It can be rationalized as a coil-like attractive interaction between different bits of the surface. Alternatively, one can derive it from the surface tension energy:

$$-\nu \nabla^2 h = -\frac{\delta \mathcal{V}\left[h\right]}{\delta h},$$

with the potential given by some constant times the surface area (through which the surface interacts with its environment). The area depends on the gradient of the height function, because such a gradient measures the faliure of the surface to align with the xy-plane. One can locally rotate the coordinates in such a way that the gradient is aligned with the axis: $\partial_y h = 0$. xh-plane, one sees a triangle whose hypothenuse is length of an inifitesimal bit of area. This hypotenuse is just $\sqrt{dx^2 + dh^2} = dx\sqrt{1 + \left(\dfrac{\partial h}{\partial x}\right)^2}$, and in rotating back to general coordinates one has

$$\mathcal{V}\left[h\right] = \nu \int dxdy \sqrt{1 + \left(\boldsymbol{\nabla} h\right)^2} = \text{const} + \frac{\nu}{2} \int d^2x \left(\boldsymbol{\nabla} h\right)^2 + \mathcal{O}\left(\left(\boldsymbol{\nabla} h\right)^4\right). \tag{4}$$

The term supposed to be responsible to the superdiffusive scaling is the growth term. It is typically said that "growth in the KPZ equation is in a direction locally normal to the surface." However, *growth can only take place through the normal direction*—a surface cannot grow along itself. The precise statement is that the total rate of growth (in the normal direction) is constant. This can be easily motivated—the normal is the only direction *locally* defined by the surface (the growth doesn't "know" where the center of the bulk is, and thus which direction corresponds to the "height"), and anything it might depend on is protected (e.g., the height by trasnlational symmetry, the direction of the normal by rotational symmetry). All in all, one has

$$v_{\text{growth}}\left(h, \boldsymbol{\nabla} h, ...\right) = \boldsymbol{v_0} \cdot \hat{\boldsymbol{h}} = v_0 \cos\Theta\left(h, \boldsymbol{\nabla} h\right).$$

with Θ the angle between the height and the normal. The cosine is, again, mostly easily visualized in coordinates where the gradient aligns with the x. In this case, in the xh-plane, since the normal is perpendicular to the tangent, the angle between the normal and the height is the same as the angle between the tangent and the x-axis. The cosine is thus

$$\cos\Theta = \frac{dx^2}{\sqrt{1 + dh^2}} = \frac{1}{\sqrt{1 + \left(\dfrac{\partial h}{\partial x}\right)^2}}.$$

Returning to the general frame of reference and expanding for small gradient (appropriate in the long wavelength regime), one finds

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + v_0 - \frac{v_0}{2}\left(\boldsymbol{\nabla} h\right)^2 + \eta\left(\boldsymbol{x}, t\right),$$

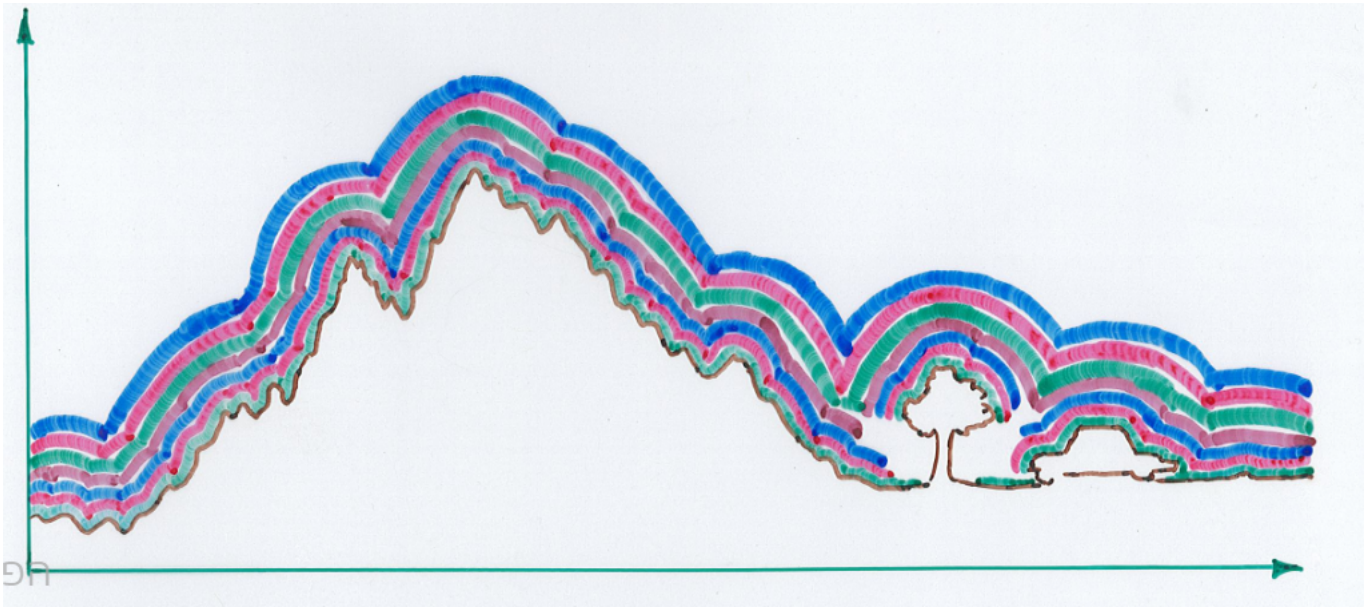

Figure 2: Kardar's illustration of the KPZ growth pattern in the saddle point approximation. Extracted from [4].

for some λ. The constant term can be eliminated by boosting the height to a comoving frame of reference $h \to h - v_0 t$, and defining $\lambda \equiv \dfrac{v_0}{2}$ one ultimately obtains

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + v_0 - \lambda \left(\boldsymbol{\nabla} h\right)^2 + \eta \left(\boldsymbol{x}, t\right). \tag{5}$$

Even before analyzing eq. (5) in detail, it is qualitatively clear why the growth term enhances the spreading the fluctuations. The more different the height function is between two point, the farther apart will normal growth send them.

While the growth term admits a simple interpretation, it cannot be derived from any potential. One way to see it is to discretize space, such that a functional derivative with respect to a potential becomes a finite-dimensional gradient $\dfrac{\partial V}{\partial h_{x,y}}$. Then one can show that the growth term has a "curl" $\dfrac{\partial v_{x,y}}{\partial h_{x',y'}} - \dfrac{\partial v_{x',y'}}{\partial h_{x,y}}$ which does not vanish in the continuum limit, so it is not a gradient.

Incidentally, the sign of the growth term is immaterial. If the sign is positive rather than negative, this simply means that the surface grows "downward" rather than "upward," and this distinction is arbitrary because one can also flip the sign of h.

3 The Many Faces of the KPZ Equation

3.1 The Schrodinger Represenation

Equation (5) has the same form of a more familiar equation: the quantum Hamiltonian-Jacobi equation for the complex phase

$$\frac{\partial S}{\partial t} = \frac{\left(\boldsymbol{\nabla} S\right)^2}{2m} - \frac{i\hbar}{2m} \nabla^2 S + V\left(\boldsymbol{x}, t\right).$$

$\partial_t S$ is the classical energy of a particle, while $\boldsymbol{\nabla} S$ is the classical momentum—hence, the first term on the right-hand side is just the kinetic energy. The second term is quantum mechanical. It couples between different classical trajectories, and is necessary for ensuring that the evolution of the wavefunction $\psi \equiv e^{iS/\hbar}$ is linear. By analogy, defining the "wavefuntion"

$$W \equiv \exp\left(-\frac{\lambda}{2\nu}h\right), \tag{6}$$

one obtains a linear equation which looks like an imaginary-time Schrodinger equation in a radnom potenital

$$\frac{\partial W}{\partial t} = \nu\nabla^2 W - \frac{\lambda}{2\nu}\eta\left(\boldsymbol{x},t\right)W. \tag{7}$$

While eq. (7) is linear, this comes at a cost: the noise doesn't merely couple to the "wavefunction" as an external source. Nevertheless, in the absence of noise eq. (7) is just a diffusion equation and be readily solved to give

$$W\left(\boldsymbol{x},t\right) \propto \int d^3x' W_0\left(\boldsymbol{x}'\right)\exp\left(-\frac{\left(\boldsymbol{x}-\boldsymbol{x}'\right)^2}{2\nu t}\right). \tag{8}$$

Taking the logarithm in the limit that the Gaussians are far apart so only one Gaussian dominates at each position (in the quantum analogue, this is a semi-classical limit), one has

$$h\left(\boldsymbol{x},t\right) = \max_{\boldsymbol{x'}}\left\{h_0\left(\boldsymbol{x}'\right) - \frac{\left(\boldsymbol{x}-\boldsymbol{x'}\right)^2}{4\lambda t}\right\}. \tag{9}$$

The height at $\boldsymbol{x}$ can be dominated by a peak at $\boldsymbol{x'}$—but only if the "benefit" exceeds the "cost" of $\frac{\left(\boldsymbol{x}-\boldsymbol{x'}\right)^2}{4\lambda t}$. The cost becomes increasingly low with time, and hence the picutre is one of parbolas emergerging from each peak in the initial height profile, gradually spreading and overrunning smaller peaks. One might think that eventually only one parabola dominates—but the cost remains infinite for large separation. This is similar to the profiles observed in experiment (see figure 1), and illustrated in a lovely figure from Kardar's lecture, reprinted in figure 2 below.

If h is initally dominated by noise (as suggesting by the fact that it doesn't scale with height), then $h_0\left(\boldsymbol{x}'\right) \sim \delta x^{1/2}$ and by eq. (9) $\delta t \sim \delta x^{3/2}$.

3.2 The Hydrodynamic Represenation

Taking the analogy with the Hamilton-Jacobi equation one step further, one might think of $\lambda\boldsymbol{\nabla}h$ as an analogue of a "flow" and of eq. (5) as a generalized equation for the "velocity potential," albeit for compressible and viscid flow. Then, taking the graident of eq. (5), one obtains a Navier-Stokes like equation with random pressures

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = \nu\nabla^2\boldsymbol{v} - \lambda\boldsymbol{\nabla}\eta\left(\boldsymbol{x},t\right). \tag{10}$$

It is seen how the same model can describe such vastly different phenomena—it can be equivalently written as a surface growth equation, a Schrodinger-like equation and a flow equation!

However, note that eq. (10) is not quite the usual Navier-Stokes equation—while the flow is generally compressible (usually, it is the pressure that enforces the constraint of incompressibility,

but here the pressure is random). Further, the flow is irrotational since it is the gradient of a potential—this is a constraint on the initial conditions, and some contraint must obviously be present since $\boldsymbol{v}$ is derived from a single scalar field and as such does not genuinely have three degrees of freedom.

Equation (10) has an important symmetry far less manifest in its original form: Galilean symmetry. The left hand side is just a derivative for a particle moving along with the flow (i.e., a time deriative in a frame of reference where the fluid is locally at rest), and it is physically clear that it remains unchanged if one moves relative to the fluid. Hence, the only way to scale $h, \boldsymbol{x}$ and t respecting Galilean symmetry is such that $\boldsymbol{\nabla} h$ scales as velocity—in other words, h scales as space squared over time. Once agan, it is seen that if h has a noise-like scaling $\delta x^{1/2}$, it follows that $\delta t \sim \delta x^{3/2}$.

4 Solving the KPZ Equation

4.1 1D: Exact Solution

The stochastic equation for the height can be transltaed into a deterministic equation for the probability. So to speak, this is analogous to the transition from the Heisenberg picture (dynamical position, static amplitudes of initial position) to the Schrodinger picture (dynamical amplitudes, static positions). This can be done by observing that $\dot{h} + \lambda \left(\boldsymbol{\nabla} h\right)^2 - \nu \nabla^2 h$ is a white noise and thus has an uncorrelated Gaussian distribution for a given history. One can then sum the result over all histories with given boundary conditions and differentiate with respect to time to obtain an evolution equation for probabilities. However, more simply, the result is that probability is conserved, with a flux given by the sum of:

1. A noisy flux, giving an equal probability for any small change in $h\left(\boldsymbol{x}\right), D\dfrac{\delta^2}{\delta h\left(\boldsymbol{x}\right)^2}\mathcal{P}\left[h\right]$, a "diffusion in the space of field configurations";

2. A deterministic flux, given simply by the probability of a configuration times the deterministic velocity in the configuration, $\mathcal{P}\left[h\right]\left(-\nu\nabla^2 h\left(\boldsymbol{x}\right) + \lambda\left(\boldsymbol{\nabla} h\left(\boldsymbol{x}\right)\right)^2\right)$, a "drift in the space of field configruations."

This is just the continuum limit of the Fokker-Planck equation for a system of particles, treating each point on the surface as a particle.

In the absence of growth, the velocity is conservative, and so the two fluxes can be cancelled at each point separately—a detailed balance—hence, the system can thermalize. The probability is, unsurprisingly, the Boltzmann weight for a set of harmonic oscillators

$$\mathcal{P}_0\left[h\right] \propto \exp\left[-\frac{\nu}{2D}\int d^d x \left(\boldsymbol{\nabla} h\right)^2\right]. \tag{11}$$

This is all quite trivial. The reason why I write all of this is because in 1D, this turns out by direct calculation to be an exact steady-state *in the presence of noise*—although no thermal equilibrium is ever reached! While the flux is nonzero, somehow the sum over fluxes to and from a given configuration cancels. In his textbook, Kardar presents it as if it is a miracle[5]. And yet, the first law of pedagogy is that there ain't no such thing as miracles, and that simple results have simple explanations. I thus decided that if I fail to provide a conceptual explanation for this

result—one that would motivate one to consider (11) as an asantz in the first place rather than merely verifying that it works "second-handedly"—then I have achieved nothing in these lectures!

The explanation I've come up with is not enirely simple in that it requires a series of insights. However, I think every insight is simple on its own, and natural enough for one to be able to be able to realize that eq. (11) is an exact 1D solution from sratch. The total probability flux is

$$\int dx \frac{\delta}{\delta h\left(x\right)}\left\{D\frac{\delta}{\delta h\left(x\right)}\mathcal{P}_0\left[h\right]-\nu\mathcal{P}_0\left[h\right]\partial_x^2 h\left(x\right)+\lambda\mathcal{P}_0\left[h\right]\left(\partial_x h\left(x\right)\right)^2\right\}. \tag{12}$$

The first term is due to noise; the second diffusion; the third growth. Now, it shouldn't surprise anyone that the $\mathcal{P}_0\left[h\right]$ makes the noise flux and the diffusion flux cancel—after all, it is an equilibrium distribution for diffusion; *it has been devised to achieve this cancellation*. Hence, the only mystery is that for some reason $\mathcal{P}_0\left[h\right]$ is conserved along the growth velocity $\lambda\left(\partial_x h\right)^2$.

Ultimately, this is not so much of an accident; it owes to the fact that there are *infinitely many* conservation laws the growth-only 1D dynamics

$$\partial_t h=-\lambda\left(\partial_x h\right)^2. \tag{13}$$

To see this, since everything (including $\mathcal{P}_0$) depends on nothing but gradients, it is natural to transform to the hydrodynamic represenation:

$$D_t v\equiv\partial_t v+v\partial_x v=0. \tag{14}$$

Eq. (14) simply states that velocity is contant along flow-lines. Therfore, so is every function of velocity

$$D_t f\left(v\right)=0. \tag{15}$$

So far all of the above readily generalizes to higher dimensions. However, eq. (15), by itself, *is not a conservation law*—constancy of a quantity along a single flow line does not imply that it is conserved, because there can be more flow lines *into a point* than *out of it* (or vice versa), and the total change in the quantity depends on points.

In a general dimension, the change in a number of flow lines (namely, sources) is given by $f\left(\boldsymbol{v}\right)\boldsymbol{\nabla}\cdot\boldsymbol{v}$ while the the change of $f\left(\boldsymbol{v}\right)$ along a given flow line is given by $\boldsymbol{v}\cdot\boldsymbol{\nabla}f\left(\boldsymbol{v}\right)$ (namely, convection). For f to be conserved, the two must cancel each other out. While both depend on derivatives of velocity, they generally depend on derivatives *in different directions*: convection depends on derivatives of velocity along the flow, while sources depend on the the derivatives of velocity along all directions where there is a deficit of incoming minus outgoing flux. However, *in 1D these two directions are the same since there is only one direction*. Now, simply because these rates of change are involve derivatives of velocity along the same dircetion doesn't imply that they are also the same in magnitude—and in general they are not. And yet, if one takes f to have a simple scaling behavior in v—i.e., to be a power law—then it is not particularly myserious that the two are proportional (after all, they have the same scaling).

Indeed, for a power law $f\left(v\right)=v^\gamma$, one finds

$$\overset{\text{sources}}{v^\gamma\partial_x v}=\gamma^{-1}\overset{\text{convection}}{v\partial_x v^\gamma}, \tag{16}$$

and so $\partial_t\left(v^\gamma\right)+v\partial_x\left(v^\gamma\right)$ *is* a conservation law. Since every function can of velocity be expanded in such powers, it follows that every function of velocity is a conserved quantity, albeit with

a veloicty potentially very different from the flow velocity—and this, in particular, applies to $\mathcal{P}_0[h] \propto \exp\left(-\frac{\nu}{2D\lambda^2}\int dx v^2\right)$.

Although the interpretation of (11) is straightforward in the absence of noise, and certainly involves no superdiffusive behavior, it turns out to change when growth is considered. By scaling h and x, demanding that $\mathcal{P}_0$ remains unchanged, one finds $h \sim \delta x^{1/2}$—which, as remarked in section 3.2, implies that $\delta x \sim \delta t^{2/3}$ by Galilean invariance. This seems highly redundant, seems the probabibility has no time-dependence; however, specific configurations of h may have time dependence, and this time dependence must *exactly* agree with the 2/3-law.

4.2 2D: Schematics of Renormalization Group Methods

In a general dimension, scaling space and time is not trivial—this is the case because the $(\boldsymbol{\nabla} h)^2$ interaction correlates all timescales and lengthscales. Hence, one cannot simply neglect small timescale and lengthscale degrees of freedom—*one must average over them.* The process of averaging over small timescale and small lengthscale degrees of freedom is the renormalization group. Because this is a brief lecture and because I wouldn't want it to turn into an algebra-dropping session, I will not go through any detailed RG calculations; rather, I will explain the key ideas beyond the methods employed, which, I think, is more illuminating. Hopefully, with some effort those interested will be able to carry out RG calculations themselves following the lecture.

To average over short wavelength modes, one takes the Fourier transform of (5) and splits it into a short wavelength and longwavelength part

$$h(\boldsymbol{k},t) = \begin{cases} \bar{h}(\boldsymbol{k},t) & k \leq e^{-l}\Lambda \\ \tilde{h}(\boldsymbol{k},t) & e^{-l}\Lambda < k \leq \Lambda \end{cases}. \tag{17}$$

Now, averaging the linear terms is trivial because they do not coupled short wavelength modes and long wavelength modes. To find out how eq. (5) changes under renormalization group, one must simply average the noise term

$$\lambda\left\langle\left(\boldsymbol{\nabla}\left(\bar{h}+\tilde{h}\right)\right)^2\right\rangle_{\sim} = \lambda\left(\boldsymbol{\nabla}\bar{h}\right)^2 + \lambda\boldsymbol{\nabla}\bar{h}\cdot\left\langle\boldsymbol{\nabla}\tilde{h}\right\rangle_{\sim} + \lambda\left\langle\left(\boldsymbol{\nabla}\tilde{h}\right)^2\right\rangle_{\sim}. \tag{18}$$

Clearly, the first term is just the same interaction one had before (that it does not change under the averaging process follows from the Galilean invariance argument in section 3.2). The second term is linear in $\bar{h}$, and can be nothing but a correction to the diffusion term. The third term is a source, and can be nothing but a correction to the noise. However to find the corrections, one has to know the moments of h, i.e., to know the solution for the KPZ equation!

Equation (5) would be solvalbe if not for the growth term. The basic idea of the perturbative expansion is to treat it a God-given external source at each order, then compute it again with the new solution and solve for the new resulting source:

$$\partial_t h_n - \nu\nabla^2 h_n = -\lambda\left[\boldsymbol{\nabla} h_{n-1}(\boldsymbol{x}',t')\right]^2 + \eta(\boldsymbol{x}',t'), \tag{19}$$

which is a diffusion equation with a source, solved to obtain

$$h_n(\boldsymbol{x},t) = \lambda\int_{t'<t} dt' d^2x' \frac{1}{\sqrt{4\nu(t-t')}}\exp\left(-\frac{(\boldsymbol{x}-\boldsymbol{x}')^2}{2\nu(t-t')}\right)\left\{-\lambda\left[\boldsymbol{\nabla} h_{n-1}(\boldsymbol{x}',t')\right]^2 + \eta(\boldsymbol{x}',t')\right\}. \tag{20}$$

Note that this rationalizes the fact that I imposed a short waelength cutoff in (17): to zeroth KPZ equation is nothing but a noise-driven diffusion equation, so that each gradient diverges for short distances as $\frac{1}{\delta x^{1/2}}$—as expected of an integral of noise. However, one could know that there is a cutoff beforehand, since the continuum models obviously does not carry over to atomic scales. With the Fourier transform of equation (20), the correlators of short wavelength modes can be "readily" compuuted to each order in perturbation theory.

Perturbative renormalization group suggests that the 2/3 law is not exact two dimensions; rather, $\delta t \sim \delta x^{1.6}$.

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